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1996 J. Phys. A: Math. Gen. 29 5575

(http://iopscience.iop.org/0305-4470/29/17/028)

J. Phys. A: Math. Gen. **29** (1996) 5575–5590. Printed in the UK

Algorithmic construction of O(3) chiral field equation hierarchy and the Landau–Lifshitz equation hierarchy via polynomial bundle

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Received 10 November 1995

Abstract. We investigate new polynomial hierarchies of Lax pairs which contain the polynomial pairs for the system of O(3) chiral field equations and Landau–Lifshitz equation introduced recently and give an algorithmic construction of the corresponding hierarchies of soliton equations. We compare the Landau–Lifshitz equation hierarchy obtained via a polynomial bundle with the hierarchy obtained via an elliptic bundle.

1. Introduction

It is well known that the so-called inverse scattering method (see [1]) allows one to apply different and fruitful approaches to the investigation of the class of nonlinear evolution equations called soliton equations. Their characteristic property is that they can be expressed as the compatibility condition of two linear operators L and M:

$$[L, M] = 0 \tag{1}$$

(This representation is called the Lax representation and the couple L, M is called the Lax pair.) In the recent work [2] we introduced new Lax pairs, polynomial in the spectral parameter, for two important physical systems†:

(A) The Landau–Lifshitz equation [3] (LL)

$$\boldsymbol{S}_t = \boldsymbol{S} \times \boldsymbol{S}_{xx} + \boldsymbol{S} \times R\boldsymbol{S}. \tag{2}$$

Here $\boldsymbol{S}(x, t) = (S_1(x, t), S_2(x, t), S_3(x, t))$ is a vector field depending on the spatial variable x and the time t, taking its values on the unit sphere $S^2 \subset \mathbb{R}^3$:

$$\boldsymbol{S}^2 = S_1^2 + S_2^2 + S_3^2 = 1 \tag{3}$$

R is the diagonal matrix

$$R = \mathrm{diag}(r_1, r_2, r_3) \qquad r_i > 0$$

with non-negative entries, $(R\boldsymbol{S})_i \equiv r_i S_i;\ \ i = 1, 2, 3.$ The LL equation describes perturbations propagating in a direction orthogonal to the anisotropy axis in a ferromagnet,

† In what follows we shall obtain the pairs for these systems as a consequence of a general construction and for this reason we do not present them now.

0305-4470/96/175575+16$19.50

and the boundary conditions for it arise naturally from the physical background. These boundary conditions can be expressed as follows

$$\lim_{x\to\pm\infty} \boldsymbol{S} = (0, 0, 1). \tag{4}$$

Remark. It should be mentioned that LL equation is related to a number of other systems of the classical mechanics, see[4].

(B) O(3) chiral field equations (CF)

$$\begin{aligned} &\boldsymbol{u}_t + \boldsymbol{u}_x + \boldsymbol{u} \times R\boldsymbol{v} = 0 \\ &\boldsymbol{v}_t - \boldsymbol{v}_x + \boldsymbol{v} \times R\boldsymbol{u} = 0. \end{aligned} \tag{5}$$

Here $\boldsymbol{v}, \boldsymbol{u}$ are two vector fields depending on x, t taking values on the unit sphere S^2 and $\times$ is the vector product symbol.

The system of O(3) chiral fields describes the dynamics in antiferromagnets and liquid crystals [5], and has application in quantum field theory [6].

It is well known that the Lax pairs divide into classes (hierarchies) and in every such hierarchy the first operators in the Lax pairs (those containing differentiation with respect to the spatial variable) coincide. Usually the pairs in the hierarchy have some natural ordering, for example, one can order polynomial pairs by the maximal degree of the spectral parameter in the second operator of the corresponding Lax pair (those containing differentiation with respect to time). The first nonlinear equation in the hierarchy of equations corresponding to the hierarchy of Lax pairs usually gives the name to the the whole hierarchy of equations and to the hierarchy of Lax pairs itself. Thus one speaks about the nonlinear Schrödinger equation hierarchy, the nonlinear Heisenberg equation hierarchy, and so on.

We must stress that the pairs known up to now both for the LL and CF were elliptic in the spectral parameter, [7, 8]. On the contrary, as we have mentioned, ours are polynomial in the spectral parameter.

We shall construct the polynomial hierarchy of Lax pairs explicitly for the O(3) CFS and for the LL cases, as well as the corresponding hierarchy of soliton equations. As far as we know the hierarchy for the chiral field equations has not been constructed explicitly until now. As to the hierarchy related to the Landau–Lifshitz equation, it will be interesting to compare the hierarchy of equations obtained via a polynomial bundle with the hierarchy obtained via an elliptic bundle (see [9–12]).

2. The polynomial hierarchy of Lax Pairs related to the system of O(3) chiral field equations

First of all, in order to make the calculations easier we shall introduce some notation and take into account that most of our matrices lie in the algebra $so(4)$, the algebra of 4×4 skew-symmetric matrices with complex entries. This algebra is semisimple, but not simple. Actually, $so(4)$ is split into a direct sum of two algebras, each of them isomorphic to $so(3)$. It can be verified that the splitting means that every element A of $so(4)$ has a unique representation of the following form:

$$A = \{\boldsymbol{u}\}_{\mathrm{I}} + \{\boldsymbol{v}\}_{\mathrm{II}} \tag{6}$$

where

$$\{\boldsymbol{u}\}_{\mathrm{I}} = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3 & -u_2 \\ -u_2 & -u_3 & 0 & u_1 \\ -u_3 & u_2 & -u_1 & 0 \end{pmatrix} \tag{7}$$

$$\{\boldsymbol{v}\}_{\mathrm{II}} = \begin{pmatrix} 0 & v_1 & v_2 & -v_3 \\ -v_1 & 0 & v_3 & v_2 \\ -v_2 & -v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix}. \tag{8}$$

With this notation the commutation relations are

$$\begin{aligned} &[\{\boldsymbol{x}\}_{\mathrm{I}}, \{\boldsymbol{y}\}_{\mathrm{I}}] = -2\{\boldsymbol{x} \times \boldsymbol{y}\}_{\mathrm{I}} \\ &[\{\boldsymbol{x}\}_{\mathrm{II}}, \{\boldsymbol{y}\}_{\mathrm{II}}] = -2\{\boldsymbol{x} \times \boldsymbol{y}\}_{\mathrm{II}} \\ &[\{\boldsymbol{x}\}_{\mathrm{I}}, \{\boldsymbol{y}\}_{\mathrm{II}}] = 0. \end{aligned} \tag{9}$$

This proves that $so(4)$ is the direct sum of two $so(3)$ algebras. There are, however, some more interesting properties of the above splitting. If J is the diagonal matrix

$$J = \begin{pmatrix} -j_1 - j_2 + j_3 & 0 & 0 & 0 \\ 0 & -j_1 + j_2 - j_3 & 0 & 0 \\ 0 & 0 & j_1 - j_2 - j_3 & 0 \\ 0 & 0 & 0 & j_1 + j_2 + j_3 \end{pmatrix} \tag{10}$$

then we have

$$\begin{aligned} &\{\boldsymbol{x}\}_{\mathrm{I}} J \{\boldsymbol{y}\}_{\mathrm{II}} - \{\boldsymbol{y}\}_{\mathrm{II}} J \{\boldsymbol{x}\}_{\mathrm{I}} = 2\left(\{\boldsymbol{x} \times K\boldsymbol{y}\}_{\mathrm{I}} + \{K\boldsymbol{x} \times \boldsymbol{y}\}_{\mathrm{II}}\right) \\ &\{\boldsymbol{x}\}_{\mathrm{I}} J \{\boldsymbol{y}\}_{\mathrm{I}} - \{\boldsymbol{y}\}_{\mathrm{I}} J \{\boldsymbol{x}\}_{\mathrm{I}} = -2\{K(\boldsymbol{x} \times \boldsymbol{y})\}_{\mathrm{II}} \\ &\{\boldsymbol{x}\}_{\mathrm{II}} J \{\boldsymbol{y}\}_{\mathrm{II}} - \{\boldsymbol{y}\}_{\mathrm{II}} J \{\boldsymbol{x}\}_{\mathrm{II}} = -2\{K(\boldsymbol{x} \times \boldsymbol{y})\}_{\mathrm{I}}. \end{aligned} \tag{11}$$

where $K = \mathrm{diag}(j_1, j_2, j_3)$ and the notation $(K\boldsymbol{z})_i \equiv j_i z_i$ is used.

Let us consider the hierarchy of Lax pairs having the following form:

$$L \equiv \frac{\partial}{\partial x} - U \qquad M_N \equiv \frac{\partial}{\partial t} - V_N \tag{12}$$

$$\begin{aligned} &U(\lambda) = \tfrac{1}{2} A(\lambda + J) \\ &V_N(\lambda) = \tfrac{1}{2}(\lambda^N B_0 + \lambda^{N-1} B_1 + \ldots + B_N)(\lambda + J) \end{aligned} \tag{13}$$

where

$$\begin{aligned} &A = \{\boldsymbol{u}\}_{\mathrm{I}} + \{\boldsymbol{v}\}_{\mathrm{II}} \\ &B_n = \{\boldsymbol{b}_n\}_{\mathrm{I}} + \{\boldsymbol{c}_n\}_{\mathrm{II}}. \end{aligned} \tag{14}$$

Remark. One can see that these pairs are natural generalizations of the 4×4 pairs we obtained in [2]. Strictly speaking from the beginning we obtained the pairs for LL equation and for the CFS in 6×6 form and then making use of the classical isomorphism between $so(3, 3)$ and $sl(4)$ cast them into 4×4 form. In the present work we prefer the 4×4 form which is simpler.

The compatibility condition between the operators L and M_N gives the following matrix equation, which must be satisfied for arbitrary λ:

$$U_t - (V_N)_x + [U, V_N] = 0. \tag{15}$$

The left-hand side of this equation is a polynomial in the spectral parameter λ and therefore all the coefficients of this polynomial must be equal to zero. This gives us the following relations:

$$\begin{aligned} &[A, B_0] = 0 \\ &[A, B_{n+1}] = 2(B_n)_x - (AJB_n - B_nJA) \qquad n = 0, 1, \ldots, N-1 \\ &2A_t + AJB_N - B_NJA - 2(B_N)_x = 0. \end{aligned} \tag{16}$$

In order to obtain the evolution equation corresponding to the Lax pair $\{L, M_N\}$ one must be able to resolve recursively the first $N+1$ of these equations and to be able to insert the result into the last equation. Making use of the particular form of the matrices A and B_n we readily arrive at the following chain of relations:

$$\begin{aligned} &\boldsymbol{u} \times \boldsymbol{b}_0 = 0 \qquad \boldsymbol{v} \times \boldsymbol{c}_0 = 0 \\ &\left.\begin{aligned} \boldsymbol{u} \times \boldsymbol{b}_{n+1} &= -(\boldsymbol{b}_n)_x - K(\boldsymbol{v} \times \boldsymbol{c}_n) + \boldsymbol{u} \times K(\boldsymbol{c}_n) - \boldsymbol{b}_n \times K(\boldsymbol{v}) \\ \boldsymbol{v} \times \boldsymbol{c}_{n+1} &= -(\boldsymbol{c}_n)_x - K(\boldsymbol{u} \times \boldsymbol{b}_n) + K(\boldsymbol{u}) \times \boldsymbol{c}_n - K(\boldsymbol{b}_n) \times \boldsymbol{v} \end{aligned}\right\} \qquad n = 0, 1, \ldots, N-1. \end{aligned} \tag{17}$$

If we are looking for an infinite set of Lax pairs then we have an infinite system of equations. We shall call this system the O(3) CF chain system. Below, in order to simplify the solution of the chain system and also for the reason that the conditions below must hold for the O(3) chiral field equations, we shall assume that

$$\boldsymbol{u}^2 = 1 \qquad \boldsymbol{v}^2 = 1. \tag{18}$$

Every solution of the O(3) CF chain system allows us to obtain the system of evolution equations

$$\left.\begin{aligned} \boldsymbol{u}_t &= (\boldsymbol{b}_N)_x + K(\boldsymbol{v} \times \boldsymbol{c}_N) - \boldsymbol{u} \times K(\boldsymbol{c}_N) + \boldsymbol{b}_N \times K(\boldsymbol{v}) \\ \boldsymbol{v}_t &= (\boldsymbol{c}_N)_x + K(\boldsymbol{u} \times \boldsymbol{b}_N) - K(\boldsymbol{u}) \times \boldsymbol{c}_N + K(\boldsymbol{b}_N) \times \boldsymbol{v} \end{aligned}\right\} \qquad N = 0, 1, 2, \ldots . \tag{19}$$

Using the next terms of the hierarchy one can also write down these equations in the form

$$\left.\begin{aligned} \boldsymbol{u}_t &= -\boldsymbol{u} \times \boldsymbol{b}_{N+1} \\ \boldsymbol{v}_t &= -\boldsymbol{v} \times \boldsymbol{c}_{N+1} \end{aligned}\right\} \qquad N = 0, 1, 2, \ldots . \tag{20}$$

Then it is clear that the constraints $\boldsymbol{u}^2 = 1$, $\boldsymbol{v}^2 = 1$ are compatible with the evolution.

The solution of the first equations in the chain is clear:

$$\boldsymbol{b}_0 = \epsilon \boldsymbol{u} \qquad \boldsymbol{c}_0 = \mu \boldsymbol{v} \tag{21}$$

where ϵ, μ are arbitrary scalar functions. The corresponding evolution equations are

$$\begin{aligned} \boldsymbol{u}_t &= \epsilon \boldsymbol{u}_x + \epsilon_x \boldsymbol{u} + (\epsilon - \mu)(\boldsymbol{u} \times K(\boldsymbol{v})) \\ \boldsymbol{v}_t &= \mu \boldsymbol{v}_x + \mu_x \boldsymbol{v} - (\epsilon - \mu)(\boldsymbol{v} \times K(\boldsymbol{u})). \end{aligned} \tag{22}$$

However, in order to resolve the next equation of the chain system (or in order to obtain evolution equations compatible with the constraints) the left-hand sides of these equations must be orthogonal to the vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. This readily gives that

ϵ, μ must be some parameters that do not depend on x. For an appropriate choice of the parameters ϵ, μ and j_i, $i = 1, 2, 3$ we obtain the O(3) CF.

At each step the situation is similar: the solution of the Nth equation is not unique and the freedom can be used to ensure that the next equation in the hierarchy can be resolved. However, some freedom still exists, but it disappears if we can fix the values of the fields and their x derivatives at some point. We shall assume that

$$\begin{aligned}
&\lim_{x\to\pm\infty} \boldsymbol{u} = \boldsymbol{u}_0 = \text{constant} \\
&\lim_{x\to\pm\infty} \boldsymbol{v} = \boldsymbol{v}_0 = \text{constant} \\
&\left.\begin{aligned}
&\lim_{x\to\pm\infty} \left(\frac{\partial}{\partial x}\right)^n \boldsymbol{u} = 0 \\
&\lim_{x\to\pm\infty} \left(\frac{\partial}{\partial x}\right)^n \boldsymbol{v} = 0
\end{aligned}\right\} \qquad n = 1, 2, \ldots\,.
\end{aligned} \tag{23}$$

These requirements are by no means necessary in order to find the hierarchy of Lax pairs and the corresponding hierarchy of evolution equations. If we choose other ones we can obtain the corresponding hierarchy in just the same way as will be done below.

As we shall need it let us briefly outline the properties of the linear operator $P_S : \mathbb{C}^3 \to \mathbb{C}^3$ defined by

$$P_S(\xi) = \boldsymbol{S} \times \xi \qquad \boldsymbol{S}^2 = 1 \tag{24}$$

$\boldsymbol{S}$ being some fixed vector. It is readily seen that the linear space $\mathbb{C}^3$ can be split into the following sum of linear subspaces:

$$\mathbb{C}^3 = \ker P_S \oplus \operatorname{im} P_S. \tag{25}$$

These subspaces are orthogonal with respect to the scalar product

$$\langle \xi, \eta \rangle = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 \tag{26}$$

and

$$\ker P_S = \mathbb{C}\boldsymbol{S} \qquad \operatorname{im} P_S = \{\xi : \langle \xi, \boldsymbol{S} \rangle = 0\}. \tag{27}$$

Let us denote by a superscript S the projection of a given vector $\boldsymbol{b}$ onto the subspace $\operatorname{im} P_S$, that is

$$\boldsymbol{b}^S = \boldsymbol{b} - \langle \boldsymbol{b}, \boldsymbol{S} \rangle \boldsymbol{S} = -(P_S)^2 \boldsymbol{b}. \tag{28}$$

If one has to solve for $\boldsymbol{x}$ the equation

$$\boldsymbol{S} \times \boldsymbol{x} = \boldsymbol{b}$$

then the problem has a solution if and only if the compatibility condition

$$\langle \boldsymbol{x}, \boldsymbol{S} \rangle = 0 \Leftrightarrow (\boldsymbol{b} = \boldsymbol{b}^S)$$

is satisfied. In that case all the solutions of the above equation are given by the formula

$$\boldsymbol{x} = -\boldsymbol{S} \times \boldsymbol{b} + \mu \boldsymbol{S} \tag{29}$$

where μ is a scalar parameter.

Let us now pass to the solution of the O(3) CF chain system. Here we have two equations of the same type as considered above. Let us denote by the superscripts 'u' and

'v' the projections on the vector subspaces orthogonal to the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. Let us consider the equations

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{b}_{n+1} &= -(\boldsymbol{b}_n)_x - K(\boldsymbol{v}\times\boldsymbol{c}_n) + \boldsymbol{u}\times K(\boldsymbol{c}_n) - \boldsymbol{b}_n\times K(\boldsymbol{v})\\ \boldsymbol{v}\times\boldsymbol{c}_{n+1} &= -(\boldsymbol{c}_n)_x - K(\boldsymbol{u}\times\boldsymbol{b}_n) + K(\boldsymbol{u})\times\boldsymbol{c}_n - K(\boldsymbol{b}_n)\times\boldsymbol{v}.\end{aligned} \tag{30}$$

It is evident that from this system one can uniquely determine the projections of the vector fields - $\boldsymbol{b}^u_{n+1}$ and $\boldsymbol{c}^v_{n+1}$, and the projections over the subspaces $\mathbb{C}\boldsymbol{u}$ and $\mathbb{C}\boldsymbol{v}$ respectively remains indefinite. These projections are given by

$$\boldsymbol{u}\langle\boldsymbol{b}_{n+1},\boldsymbol{u}\rangle \qquad \boldsymbol{v}\langle\boldsymbol{c}_{n+1},\boldsymbol{v}\rangle.$$

We shall find how these projections can be expressed thorough $\boldsymbol{b}^u_{n+1}$, $\boldsymbol{c}^v_{n+1}$. Let us consider the expression

$$\frac{\partial}{\partial x}\langle\boldsymbol{b}_{n+1},\boldsymbol{u}\rangle = \left\langle\frac{\partial}{\partial x}\boldsymbol{b}_{n+1},\boldsymbol{u}\right\rangle + \left\langle\boldsymbol{b}_{n+1},\frac{\partial}{\partial x}\boldsymbol{u}\right\rangle.$$

Suppose now that the $(n+2)$th equation in the chain system can be resolved. Then we can write

$$(\boldsymbol{b}_{n+1})_x = -\boldsymbol{u}\times\boldsymbol{b}_{n+2} - K(\boldsymbol{v}\times\boldsymbol{c}_{n+1}) + \boldsymbol{u}\times K(\boldsymbol{c}_{n+1}) - \boldsymbol{b}_{n+1}\times K(\boldsymbol{v}).$$

Then evidently

$$\begin{aligned}\langle\boldsymbol{u},(\boldsymbol{b}_{n+1})_x\rangle &= \langle\boldsymbol{u}\times K(\boldsymbol{v}),\boldsymbol{b}_{n+1}\rangle + \langle\boldsymbol{v}\times K(\boldsymbol{u}),\boldsymbol{c}_{n+1}\rangle\\ &= \langle\boldsymbol{u}\times K(\boldsymbol{v}),\boldsymbol{b}^u_{n+1}\rangle + \langle\boldsymbol{v}\times K(\boldsymbol{u}),\boldsymbol{c}^v_{n+1}\rangle\end{aligned}$$

and we have

$$\langle\boldsymbol{u},\boldsymbol{b}_{n+1}\rangle = \int_{\pm\infty}^{x}\left(\langle\boldsymbol{b}^u_{n+1},\boldsymbol{u}_x\rangle + \langle\boldsymbol{u}\times K(\boldsymbol{v}),\boldsymbol{b}^u_{n+1}\rangle + \langle\boldsymbol{v}\times K(\boldsymbol{u}),\boldsymbol{c}^v_{n+1}\rangle\right)\,\mathrm{d}x. \tag{31}$$

In just the same manner

$$\langle\boldsymbol{v},\boldsymbol{c}_{n+1}\rangle = \int_{\pm\infty}^{x}\left(\langle\boldsymbol{c}^v_{n+1},\boldsymbol{v}_x\rangle + \langle\boldsymbol{u}\times K(\boldsymbol{v}),\boldsymbol{b}^u_{n+1}\rangle + \langle\boldsymbol{v}\times K(\boldsymbol{u}),\boldsymbol{c}^v_{n+1}\rangle\right)\,\mathrm{d}x. \tag{32}$$

Inserting these expressions into the formulae

$$\boldsymbol{b}_{n+1} = \boldsymbol{b}^u_{n+1} + \boldsymbol{u}\langle\boldsymbol{u},\boldsymbol{b}_{n+1}\rangle \qquad \boldsymbol{c}_{n+1} = \boldsymbol{b}^v_{n+1} + \boldsymbol{v}\langle\boldsymbol{v},\boldsymbol{c}_{n+1}\rangle \tag{33}$$

we get $\boldsymbol{b}_{n+1}$, $\boldsymbol{c}_{n+1}$ expressed through the projections $\boldsymbol{b}^u_{n+1}$, $\boldsymbol{c}^v_{n+1}$. In what follows we shall assume that the projections over the subspaces $\mathbb{C}\boldsymbol{u}$ and $\mathbb{C}\boldsymbol{v}$ are given by the above formulae. Then the last thing that remains to be done is to express $\boldsymbol{b}^u_{n+1}$, $\boldsymbol{c}^v_{n+1}$ in terms of $\boldsymbol{b}^u_n$, $\boldsymbol{c}^v_n$. We formulate the final answer in the following proposition.

Proposition 1. The CF chain system has the following solution:

$$\begin{aligned}&\boldsymbol{b}_0 = \epsilon\boldsymbol{u} \qquad \boldsymbol{c}_0 = \mu\boldsymbol{v}\\ &\boldsymbol{b}^u_{n+1} = \boldsymbol{u}\times(\boldsymbol{b}^u_n)_x + \langle\boldsymbol{u},\boldsymbol{b}_n\rangle\boldsymbol{u}\times\boldsymbol{u}_x + \left[K(\boldsymbol{c}^v_n)\right]^u - (\langle\boldsymbol{u},\boldsymbol{b}_n\rangle - \langle\boldsymbol{v},\boldsymbol{c}_n\rangle)[K(v)]^u\\ &\qquad +\boldsymbol{u}\times K(\boldsymbol{v}\times\boldsymbol{c}^v_n) + \langle\boldsymbol{u},K(\boldsymbol{v})\rangle\boldsymbol{b}^u_n\\ &\boldsymbol{c}^v_{n+1} = \boldsymbol{v}\times(\boldsymbol{c}^v_n)_x + \langle\boldsymbol{v},\boldsymbol{c}_n\rangle\boldsymbol{v}\times\boldsymbol{v}_x + \left[K(\boldsymbol{b}^u_n)\right]^v + (\langle\boldsymbol{u},\boldsymbol{b}_n\rangle - \langle\boldsymbol{v},\boldsymbol{c}_n\rangle)[K(u)]^v\\ &\qquad +\boldsymbol{v}\times K(\boldsymbol{u}\times\boldsymbol{b}^u_n) + \langle\boldsymbol{u},K(\boldsymbol{v})\rangle\boldsymbol{c}^v_n\\ &\qquad\qquad n = 0,1,2,\ldots\end{aligned} \tag{34}$$

where ϵ, μ are arbitrary constants and

$$\begin{aligned}\langle \boldsymbol{u}, \boldsymbol{b}_n\rangle &= \int_{\pm\infty}^{x} \left(\langle \boldsymbol{b}_n^u, \boldsymbol{u}_x\rangle + \langle \boldsymbol{u}\times K(\boldsymbol{v}), \boldsymbol{b}_n^u\rangle + \langle \boldsymbol{v}\times K(\boldsymbol{u}), \boldsymbol{c}_n^v\rangle\right)\, \mathrm{d}x\\ \langle \boldsymbol{v}, \boldsymbol{c}_n\rangle &= \int_{\pm\infty}^{x} \left(\langle \boldsymbol{c}_n^v, \boldsymbol{v}_x\rangle + \langle \boldsymbol{u}\times K(\boldsymbol{v}), \boldsymbol{b}_n^u\rangle + \langle \boldsymbol{v}\times K(\boldsymbol{u}), \boldsymbol{c}_n^v\rangle\right)\, \mathrm{d}x.\end{aligned} \tag{35}$$

One can see that the couple of functions $(\boldsymbol{b}_{n+1}^u, \boldsymbol{c}_{n+1}^v)$ is expressed through the couple $(\boldsymbol{b}_n^u, \boldsymbol{c}_n^v)$ with the help of some integro-differential operator $\boldsymbol{A}_\pm(\boldsymbol{u}, \boldsymbol{v})$, depending on $\boldsymbol{u}(x), \boldsymbol{v}(x)$. The choice of subscript '+' or '−' corresponds to the choice of $+\infty$ or $-\infty$ for the integration limit in the corresponding expressions. In other words

$$\begin{pmatrix} \boldsymbol{b}_{n+1}^u \\ \boldsymbol{c}_{n+1}^v \end{pmatrix} = \boldsymbol{A}_\pm(\boldsymbol{u}, \boldsymbol{v}) \begin{pmatrix} \boldsymbol{b}_n^u \\ \boldsymbol{c}_n^v \end{pmatrix}. \tag{36}$$

We shall not write the explicit formula for $\boldsymbol{A}_\pm$ as it is too complicated and besides it can easily be derived from the above proposition. The operators that allow one to obtain the hierarchies of soliton equations recursively are called recursion operators or generating operators. It turns out that their existence is very important. For example, they play a crucial role in describing the hierarchies of Hamiltonian structures for the soliton equations. Other important application of the generating operators is that their spectral decomposition allows one to obtain the so-called expansions over the squared (or adjoint) solutions which proved to be a very useful tool in the investigation of soliton equations (see, for example, [14], where the case of the generating operator for the hierarchy of Heisenberg ferromagnet is considered).

It will be superfluous to discuss the hierarchies of Hamiltonian structures for the CF here and so we shall leave this kind of question for a future publication. We remark only that as far as we know the generating operator for the CF hierarchy has been calculated in [11], but from a quite different background, namely using the fact that this operator gives the relation between compatible Poisson tensors defined via an elliptic bundle. However, the hierarchies of Lax pairs were not obtained in [11].

At the end of this section let us write the first two systems of the CF hierarchy.

1. $N = 0$. The first equation in the CF hierarchy:

$$\begin{aligned}\boldsymbol{u}_t &= \epsilon \boldsymbol{u}_x + (\epsilon - \mu)(\boldsymbol{u}\times K(\boldsymbol{v}))\\ \boldsymbol{v}_t &= \mu \boldsymbol{v}_x - (\epsilon - \mu)(\boldsymbol{v}\times K(\boldsymbol{u})).\end{aligned} \tag{37}$$

After the following choice of the parameters: $\epsilon = -1$, $\mu = 1$, $R = 2K$ and changing $\boldsymbol{u}$ to $-\boldsymbol{u}$ we obtain the O(3) chiral field equations system (5).

2. $N = 1$. The second equation in the CF hierarchy:

$$\begin{aligned}\boldsymbol{u}_t &= \epsilon \boldsymbol{u}\times \boldsymbol{u}_{xx} + 2\epsilon(\langle \boldsymbol{u}, K(\boldsymbol{v})\rangle - \langle \boldsymbol{u}_0, K(\boldsymbol{v}_0)\rangle)\boldsymbol{u}_x - \epsilon K(\boldsymbol{v}_x) + \epsilon\langle \boldsymbol{u}, K(\boldsymbol{v}_x)\rangle \boldsymbol{u}\\ &\quad -\mu \boldsymbol{u}\times K(\boldsymbol{v}\times \boldsymbol{v}_x) + (\epsilon-\mu)(\langle \boldsymbol{u}, K(\boldsymbol{v})\rangle - \langle \boldsymbol{u}_0, K(\boldsymbol{v}_0)\rangle)(\boldsymbol{u}\times K(\boldsymbol{v}))\\ &\quad -(\epsilon-\mu)\boldsymbol{u}\times K^2(\boldsymbol{u})\\ \boldsymbol{v}_t &= \mu \boldsymbol{v}\times \boldsymbol{v}_{xx} + 2\mu(\langle \boldsymbol{v}, K(\boldsymbol{u})\rangle - \langle \boldsymbol{v}_0, K(\boldsymbol{u}_0)\rangle)\boldsymbol{v}_x - \mu K(\boldsymbol{u}_x) + \mu\langle \boldsymbol{v}, K(\boldsymbol{u}_x)\rangle \boldsymbol{v}\\ &\quad -\epsilon \boldsymbol{v}\times K(\boldsymbol{u}\times \boldsymbol{u}_x) - (\epsilon-\mu)(\langle \boldsymbol{v}, K(\boldsymbol{u})\rangle - \langle \boldsymbol{v}_0, K(\boldsymbol{u}_0)\rangle)(\boldsymbol{v}\times K(\boldsymbol{u}))\\ &\quad +(\epsilon-\mu)\boldsymbol{v}\times K^2(\boldsymbol{v}).\end{aligned} \tag{38}$$

An interesting special case of this system is obtained for $\mu = 0$. Then we have

$$\begin{aligned} \boldsymbol{u}_t &= \epsilon \boldsymbol{u} \times \boldsymbol{u}_{xx} + 2\epsilon(\langle \boldsymbol{u}, K(\boldsymbol{v})\rangle - \langle \boldsymbol{u}_0, K(\boldsymbol{v}_0)\rangle)\boldsymbol{u}_x - \epsilon K(\boldsymbol{v}_x) + \epsilon\langle \boldsymbol{u}, K(\boldsymbol{v}_x)\rangle \boldsymbol{u} \\ &\quad + \epsilon(\langle \boldsymbol{u}, K(\boldsymbol{v})\rangle - \langle \boldsymbol{u}_0, K(\boldsymbol{v}_0)\rangle)(\boldsymbol{u} \times K(\boldsymbol{v})) - \epsilon \boldsymbol{u} \times K^2(\boldsymbol{u}) \\ \boldsymbol{v}_t &= -\epsilon \boldsymbol{v} \times K(\boldsymbol{u} \times \boldsymbol{u}_x) - \epsilon(\langle \boldsymbol{v}, K(\boldsymbol{u})\rangle - \langle \boldsymbol{v}_0, K(\boldsymbol{u}_0)\rangle)(\boldsymbol{v} \times K(\boldsymbol{u})) + \epsilon \boldsymbol{v} \times K^2(\boldsymbol{v}). \end{aligned} \tag{39}$$

Another reduction of the general system (38) is obtained if we assume that $\epsilon = \mu$. Then we have

$$\begin{aligned} \boldsymbol{u}_t &= \epsilon \boldsymbol{u} \times \boldsymbol{u}_{xx} + 2\epsilon(\langle \boldsymbol{u}, K(\boldsymbol{v})\rangle - \langle \boldsymbol{u}_0, K(\boldsymbol{v}_0)\rangle)\boldsymbol{u}_x - \epsilon K(\boldsymbol{v}_x) + \epsilon\langle \boldsymbol{u}, K(\boldsymbol{v}_x)\rangle \boldsymbol{u} \\ &\quad - \epsilon \boldsymbol{u} \times K(\boldsymbol{v} \times \boldsymbol{v}_x) \\ \boldsymbol{v}_t &= \epsilon \boldsymbol{v} \times \boldsymbol{v}_{xx} + 2\epsilon(\langle \boldsymbol{v}, K(\boldsymbol{u})\rangle - \langle \boldsymbol{v}_0, K(\boldsymbol{u}_0)\rangle)\boldsymbol{v}_x - \epsilon K(\boldsymbol{u}_x) + \epsilon\langle \boldsymbol{v}, K(\boldsymbol{u}_x)\rangle \boldsymbol{v} \\ &\quad - \epsilon \boldsymbol{v} \times K(\boldsymbol{u} \times \boldsymbol{u}_x). \end{aligned} \tag{40}$$

The next systems of the hierarchy can also be obtained without much difficulties but for them there is less hope for any physical applications.

3. Polynomial hierarchy of Lax Pairs related to the Landau–Lifshitz equation

The Landau–Lifshitz equation can be obtained within the general scheme described above if instead of the constraint $\boldsymbol{v}^2 = 1$ we impose the constraint $\boldsymbol{v} = 0$. Unfortunately, as the condition $\boldsymbol{v}^2 = 1$ was essential in all our constructions, one cannot simply insert $\boldsymbol{v} = 0$ in the solution for the O(3) CF chain system in order to obtain the solution for the corresponding chain system for LL equation.

Remark. It is not difficult to check that if instead of the constraints $\boldsymbol{v} = 0, \boldsymbol{u}^2 = 1$ we choose the constraints $\boldsymbol{u} = 0, \boldsymbol{v}^2 = 1$ then we shall obtain the same hierarchy of Lax pairs. Thus in all the constructions there exists a symmetry between the two so(3) subalgebras in so(4).

In order to obtain the LL equation in the same terms as it was introduced, we shall change the notation and in what follows shall put $\boldsymbol{u} \equiv \boldsymbol{S}$. Then the chain relations are reduced to

$$\begin{aligned} &\boldsymbol{S} \times \boldsymbol{b}_0 = 0 \\ &\left.\begin{aligned} &\boldsymbol{S} \times \boldsymbol{b}_{n+1} = -(\boldsymbol{b}_n)_x + \boldsymbol{S} \times K(\boldsymbol{c}_n) \\ &(\boldsymbol{c}_n)_x = -K(\boldsymbol{S} \times \boldsymbol{b}_n) + K(\boldsymbol{S}) \times \boldsymbol{c}_n \end{aligned}\right\} \qquad n = 0, 1, \ldots, N-1. \end{aligned} \tag{41}$$

We shall refer the above system of equations as the LL chain system.

The corresponding hierarchy of evolution equations is then

$$\boldsymbol{S}_t = (\boldsymbol{b}_N)_x - \boldsymbol{S} \times K(\boldsymbol{c}_N) \qquad N = 0, 1, 2, \ldots \tag{42}$$

or using the next term in the hierarchy we can write

$$\boldsymbol{S}_t = -\boldsymbol{S} \times \boldsymbol{b}_{N+1} \qquad N = 0, 1, 2, \ldots\ . \tag{43}$$

Thus, as was for the case for the O(3) CF the Nth evolution equation obeys the constraint $\boldsymbol{S}^2 = 1$ if the $(N+1)$th relation in the chain can be resolved. Therefore, if we can show that there exist solution of the infinite system defined above, then all the evolution equations will respect automatically the constraint. To begin with one must make a choice for the first terms. We shall consider the case

$$\boldsymbol{b}_0 = \boldsymbol{S} \qquad \boldsymbol{c}_0 = 0 \tag{44}$$

as it leads directly to the LL equation. The general case $\boldsymbol{b}_0 = f\boldsymbol{S}$, where f is some scalar function and $\boldsymbol{c}_0$ is a solution of the equation

$$(\boldsymbol{c}_0)_x = K(\boldsymbol{S}) \times \boldsymbol{c}_0 \tag{45}$$

seems to be more complicated, but, in fact, we must recall that in order to obtain evolution equation having the form

$$\boldsymbol{S}_t = \boldsymbol{F}(\boldsymbol{S}, \boldsymbol{S}_x, \ldots)$$

the function f and the solution $\boldsymbol{c}_0$ must depend on x, t only through $\boldsymbol{S}(x, t)$ and its derivatives. A brief analysis then shows that $\boldsymbol{c}_0 = 0$ is the only appropriate choice and the general case can be treated along the same lines.

The hierarchy of equations can be described explicitly if at each step we can present the solution of the equation

$$(\boldsymbol{c}_n)_x = -K(\boldsymbol{S} \times \boldsymbol{b}_n) + K(\boldsymbol{S}) \times \boldsymbol{c}_n. \tag{46}$$

In order to do this we shall need some preparation. Let us introduce the sequence of diagonal matrices $K^{(n)}$, $n = 1, 2, \ldots$, satisfying the relations

$$\begin{aligned}
&K^{(1)} \equiv K\\
&K^{(1)}(\boldsymbol{a}) \times K^{(1)}(\boldsymbol{b}) = K^{(2)}(\boldsymbol{a} \times \boldsymbol{b})\\
&K^{(1)}(\boldsymbol{a} \times K^{(1)}K^{(1)}(\boldsymbol{b})) + K^{(1)}(\boldsymbol{a}) \times K^{(2)}(\boldsymbol{b}) = K^{(3)}(\boldsymbol{a} \times \boldsymbol{b})\\
&K^{(1)}(\boldsymbol{a} \times K^{(1)}K^{(2)}(\boldsymbol{b})) + K^{(2)}(\boldsymbol{a} \times K^{(1)}K^{(1)}(\boldsymbol{b})) + K^{(1)}(\boldsymbol{a}) \times K^{(3)}(\boldsymbol{b}) = K^{(4)}(\boldsymbol{a} \times \boldsymbol{b})\\
&\qquad\vdots\\
&\sum_{i=1}^{n-2} K^{(i)}(\boldsymbol{a} \times K^{(1)}K^{(n-i-1)}(\boldsymbol{b})) + K^{(1)}(\boldsymbol{a}) \times K^{(n-1)}(\boldsymbol{b}) = K^{(n)}(\boldsymbol{a} \times \boldsymbol{b}) \qquad n = 3, 4, \ldots
\end{aligned} \tag{47}$$

for arbitrary choice of the vectors $\boldsymbol{a}, \boldsymbol{b}$.

Lemma. The sequence of diagonal matrices $K^{(n)}$

$$K^{(n)} = \mathrm{diag}(K_1^{(n)}, K_2^{(n)}, K_3^{(n)}) \tag{48}$$

is well defined and the entries $K_i^{(n)}, i = 1, 2, 3$ of $K^{(n)}$ are homogeneous polynomials of degree n with respect to the variables j_1, j_2, j_3.

Proof. Let us calculate the first terms of the sequence. One can easily obtain

$$\begin{aligned}
&K_1^{(1)} = j_1 \qquad K_1^{(2)} = j_2 j_3 \qquad K_1^{(3)} = j_1(j_2^2 + j_3^2) \qquad K_1^{(4)} = j_2 j_3(2j_1^2 + j_2^2 + j_3^2)\\
&K_2^{(1)} = j_2 \qquad K_2^{(2)} = j_1 j_3 \qquad K_2^{(3)} = j_2(j_1^2 + j_3^2) \qquad K_2^{(4)} = j_1 j_3(j_1^2 + 2j_2^2 + j_3^2)\\
&K_3^{(1)} = j_3 \qquad K_3^{(2)} = j_1 j_2 \qquad K_3^{(3)} = j_3(j_1^2 + j_2^2) \qquad K_3^{(4)} = j_1 j_2(j_1^2 + j_2^2 + 2j_3^2).
\end{aligned} \tag{49}$$

Since the statement of the lemma is true for $n = 1, 2, 3, 4$, one can try to prove the lemma by induction. Suppose that the sequence $K^{(n)}$ has the needed properties for $n = 1, 2, \ldots; N \geqslant 4$. Then we shall prove that there exists the unique diagonal matrix $K^{(N+1)}$ such that

$$\sum_{i=1}^{N-1} K^{(i)}(\boldsymbol{a} \times K^{(1)}K^{(s-i)}(\boldsymbol{b})) + K^{(1)}(\boldsymbol{a}) \times K^{(N)}(\boldsymbol{b}) = K^{(N+1)}(\boldsymbol{a} \times \boldsymbol{b})$$

for arbitrary choice of the vectors $\boldsymbol{a}, \boldsymbol{b}$. Let us calculate the first component of the left-hand side of this vector equality. We get

$$\left(\sum_{i=1}^{N-1} K_1^{(i)} K_3^{(1)} K_3^{(N-i)} + K_2^{(1)} K_3^{(N)}\right) a_2 b_3 - \left(\sum_{i=1}^{N-1} K_1^{(i)} K_2^{(1)} K_2^{(N-i)} + K_3^{(1)} K_2^{(N)}\right) a_3 b_2.$$

In order to write this expression in the form

$$K_1^{(N+1)}(a_2 b_3 - a_3 b_2)$$

with some coefficient $K_1^{(N+1)}$ which does not depend on $a_i, b_i; i = 1, 2, 3$ it is necessary and sufficient to have

$$W \equiv \sum_{i=1}^{N-1} K_1^{(i)} (K_3^{(1)} K_3^{(N-i)} - K_2^{(1)} K_2^{(N-i)}) + K_2^{(1)} K_3^{(N)} - K_3^{(1)} K_2^{(N)} = 0.$$

We recall that by the inductive assumption for all $2 \leqslant s \leqslant N-1$ we have

$$K_1^{(s+1)} = K_2^{(1)} K_3^{(s)} + K_3^{(1)} \sum_{i=1}^{s-1} K_1^{(i)} K_3^{(s-i)}$$

$$K_1^{(s+1)} = K_3^{(1)} K_2^{(s)} + K_2^{(1)} \sum_{i=1}^{s-1} K_1^{(i)} K_2^{(s-i)}$$

and also four other relations which can be obtained from the above ones with cyclic permutation of the indices. They correspond to the other two components of the vector relations from the lemma. Then we can write

$$\begin{aligned} W = {} & K_2^{(1)} \left(K_2^{(1)} K_1^{(N-1)} + \sum_{l=1}^{N-2} K_3^{(l)} K_1^{(N-l-1)} K_1^{(1)} \right) \\ & - K_3^{(1)} \left(K_3^{(1)} K_1^{(N-1)} + \sum_{l=1}^{N-2} K_2^{(l)} K_1^{(N-l-1)} K_1^{(1)} \right) \\ & + \sum_{i=1}^{N-1} \left(K_1^{(i)} K_3^{(N-i)} K_3^{(1)} - K_1^{(i)} K_2^{(N-i)} K_2^{(1)} \right) \\ = {} & K_3^{(1)} \left(\sum_{l=1}^{N-3} K_1^{(l)} (K_3^{(N-l)} - K_1^{(1)} K_2^{(N-l-1)}) \right) + K_3^{(1)} K_1^{(N-2)} (K_3^{(2)} - K_1^{(1)} K_2^{(1)}) \\ & - K_2^{(1)} \left(\sum_{l=1}^{N-3} K_1^{(l)} (K_2^{(N-l)} - K_1^{(1)} K_3^{(N-l-1)}) \right) \\ & - K_2^{(1)} K_1^{(N-2)} (K_2^{(2)} - K_1^{(1)} K_3^{(1)}). \end{aligned}$$

From the explicit formulae for $K^{(2)}$ it follows that the terms which are not under the summation are zero. As for the rest of the expression W, it vanishes due to the the relations

$$K_3^{(N-l)} - K_1^{(1)} K_2^{(N-l-1)} = \sum_{j=1}^{N-l-2} K_2^{(1)} K_3^{(j)} K_2^{(N-l-j-1)}$$

$$K_2^{(N-l)} - K_1^{(1)} K_3^{(N-l-1)} = \sum_{j=1}^{N-l-2} K_3^{(1)} K_3^{(j)} K_2^{(N-l-j-1)}.$$

Exactly the same procedure can be applied for the other two components. The statement that the entries of $K^{(n)}$ are homogeneous polynomials of degree n readily follows from the proof. □

Proposition 2. Suppose $\boldsymbol{b}_m, \boldsymbol{c}_m$; $m = 0, 1, \ldots, n-1 \geqslant 0$ are solutions of the chain system

$$\left.\begin{aligned} &\boldsymbol{b}_0 = \boldsymbol{S} \qquad \boldsymbol{c}_0 = 0 \\ &\boldsymbol{S} \times \boldsymbol{b}_{m+1} = -(\boldsymbol{b}_m)_x + \boldsymbol{S} \times K(\boldsymbol{c}_m) \\ &(\boldsymbol{c}_m)_x = -K(\boldsymbol{S} \times \boldsymbol{b}_m) + K(\boldsymbol{S}) \times \boldsymbol{c}_m \end{aligned}\right\} \qquad m = 1, 2, \ldots, n-1. \tag{50}$$

Then

$$\boldsymbol{c}_n = \sum_{q=1}^{n} (-1)^{q-1} K^{(q)}(\boldsymbol{b}_{n-q}) \tag{51}$$

is a solution of the equation

$$(\boldsymbol{c}_n)_x = -K(\boldsymbol{S} \times \boldsymbol{b}_n) + K(\boldsymbol{S}) \times \boldsymbol{c}_n.$$

Proof. We shall prove this proposition by induction. For $n = 1$ we have

$$-\boldsymbol{S} \times \boldsymbol{b}_1 = (\boldsymbol{b}_0)_x = (\boldsymbol{S})_x.$$

As $\boldsymbol{S}^2 = 1$, the vector $\boldsymbol{S}_x$ is orthogonal to $\boldsymbol{S}$ and one gets $\boldsymbol{b}_1 = \boldsymbol{S} \times \boldsymbol{S}_x + \alpha \boldsymbol{S}$, where α is a scalar parameter.Then it is readily seen that $\boldsymbol{c}_1 = K(\boldsymbol{S})$ solves the equation

$$(\boldsymbol{c}_1)_x = -K(\boldsymbol{S} \times \boldsymbol{b}_1) + K(\boldsymbol{S}) \times \boldsymbol{c}_1.$$

We shall now assume that the proposition is true for all $n = 1, 2, \ldots, N-1$ and shall prove it for $n = N$. In order to do this let us calculate

$$(\boldsymbol{c}_N)_x = \sum_{q=1}^{N} (-1)^{q-1} K^{(q)}((\boldsymbol{b}_{N-q})_x).$$

Taking into account that $\boldsymbol{b}_i$ are solutions of the chain system we get

$$(\boldsymbol{c}_N)_x = -\sum_{q=1}^{N} (-1)^{q-1} K^{(q)}(\boldsymbol{S} \times \boldsymbol{b}_{N-q+1}) + \sum_{q=1}^{N-1} (-1)^{q-1} K^{(q)}(\boldsymbol{S} \times K^{(1)}(\boldsymbol{c}_{N-q}))$$

where we have used that $\boldsymbol{c}_0 = 0$. Inserting into this equation the expressions for $\boldsymbol{c}_{N-q}$ we obtain

$$\begin{aligned} (\boldsymbol{c}_N)_x = &-\sum_{q=1}^{N} (-1)^{q-1} K^{(q)}(\boldsymbol{S} \times \boldsymbol{b}_{N-q+1}) \\ &+ \sum_{q=1}^{N-1} (-1)^{q-1} K^{(q)}(\boldsymbol{S} \times K^{(1)}(\sum_{r=1}^{N-q} (-1)^{r-1} K^{(r)}(\boldsymbol{b}_{N-q-r}))). \end{aligned}$$

Then

$$\begin{aligned} &(\boldsymbol{c}_N)_x + K(\boldsymbol{S} \times \boldsymbol{b}_N) - K(\boldsymbol{S}) \times \boldsymbol{c}_N \\ &\quad = \sum_{q=1}^{N-1} (-1)^{q-1} K^{(q)}\left(\boldsymbol{S} \times K^{(1)}\left(\sum_{r=1}^{N-q} (-1)^{r-1} K^{(r)}(\boldsymbol{b}_{N-q-r})\right)\right) \\ &\qquad - \sum_{q=1}^{N} (-1)^{q-1} K^{(q)}(\boldsymbol{S} \times \boldsymbol{b}_{N-q}) + K^{(1)}(\boldsymbol{S} \times \boldsymbol{b}_N) \end{aligned}$$

$$-K^{(1)}(\boldsymbol{S}) \times \sum_{q=1}^{N}(-1)^{q-1}K^{(q)}(\boldsymbol{b}_{N-q})$$

$$= \sum_{q=1}^{N-1}(-1)^{q-1}K^{(q)}\left(\boldsymbol{S} \times K^{(1)}\left(\sum_{r=1}^{N-q}(-1)^{r-1}K^{(r)}(\boldsymbol{b}_{N-q-r})\right)\right)$$

$$-\sum_{q=2}^{N}(-1)^{q-1}K^{(q)}(\boldsymbol{S} \times \boldsymbol{b}_{N-q+1}) - K^{(1)}(\boldsymbol{S}) \times \sum_{q=1}^{N}(-1)^{q-1}K^{(q)}(\boldsymbol{b}_{N-q})$$

$$= \sum_{q=2}^{N-1}(-1)^{q+1}\left(K^{(q+1)}(\boldsymbol{S} \times \boldsymbol{b}_{N-q+1}) - K^{(1)}(\boldsymbol{S}) \times K^{(q)}(\boldsymbol{b}_{N-q})\right)$$

$$-\sum_{q=2}^{N-1}(-1)^{q+1}\left(\sum_{l+r=q}K^{(l)}(\boldsymbol{S} \times K^{(1)}K^{(r)}(\boldsymbol{b}_{N-q})\right)$$

$$+(-1)^{N}\left(K^{(1)}(\boldsymbol{S}) \times K^{(q)}(\boldsymbol{b}_0) + \sum_{q+r=N}K^{(q)}(\boldsymbol{S} \times K^{(1)}K^{(r)}(\boldsymbol{b}_0)\right)$$

$$+K^{(2)}(\boldsymbol{S} \times \boldsymbol{b}_{N-1}) - K^{(1)}(\boldsymbol{S}) \times K^{(1)}(\boldsymbol{b}_{N-1}).$$

From the definition of the matrices $K^{(q)}$ it follows that the above expression is equal to

$$(-1)^{N}K^{(N+1)}(\boldsymbol{S} \times \boldsymbol{b}_0) = 0.$$

Thus $\boldsymbol{c}_N$ is a solution of the equation

$$(\boldsymbol{c}_N)_x = -K(\boldsymbol{S} \times \boldsymbol{b}_N) + K(\boldsymbol{S}) \times \boldsymbol{c}_N$$

and the proposition is proved. □

Now we know how to solve the second part of the equations in the chain system. The first part of these equations runs as follows:

$$\boldsymbol{S} \times \boldsymbol{b}_{n+1} = -(\boldsymbol{b}_n)_x + \boldsymbol{S} \times K(\boldsymbol{c}_n).$$

We have considered similar equations dealing with the CF chain system. As outlined in the previous section in order to solve for $\boldsymbol{b}_{n+1}$ the compatibility condition

$$\langle(\boldsymbol{b}_n)_x, \boldsymbol{S}\rangle = 0 \tag{52}$$

must be satisfied and to ensure it we must use the freedom in the determination of the solution for the previous equation. Let us consider the following decomposition of $\boldsymbol{b}_n$, $n \geqslant 1$:

$$\boldsymbol{b}_n = \boldsymbol{b}_n^S + \boldsymbol{S}\langle\boldsymbol{b}_n, \boldsymbol{S}\rangle. \tag{53}$$

As before when one solves the chain relations one recovers uniquely $\boldsymbol{b}_n^S$ and all the non-uniqueness appears in the determination of $\langle\boldsymbol{b}_n, \boldsymbol{S}\rangle$. They are recovered from $\boldsymbol{b}_n^S$ if in addition one can fix the values of the field $\boldsymbol{S}$ and its x-derivatives at some point. Then the calculations are exactly the same as in the case of CF chain system and we shall omit them. Before presenting the final result, however, there is one point we want to discuss. As already mentioned, in order to obtain the unique solution on each step one must fix the values of the vector field $\boldsymbol{S}$ and its x derivatives at some point of $\mathbb{R}^3$ (including infinity). We shall assume that the function $\boldsymbol{S}$ has the following property:

$$\begin{aligned}&\lim_{x\to\pm\infty} \boldsymbol{S} = \boldsymbol{S}_0 = \text{constant}\\&\lim_{x\to\pm\infty}\left(\frac{\partial}{\partial x}\right)^n \boldsymbol{S} = 0\\&n = 1, 2, \ldots .\end{aligned} \tag{54}$$

Usually, for the LL equation the condition

$$\lim_{x\to\pm\infty} \boldsymbol{S} = (0, 0, 1)$$

is imposed. We consider the above condition in order to obtain more symmetrical expressions with respect to a cyclic permutation of the indices 1, 2, 3.

Remark. The above requirements seem quite natural for the Landau–Lifshitz equation, but, of course , if one is looking only for the hierarchy of equations they are not absolutely necessary.

Thus we arrive at the following solution of the LL chain system:

$$\boldsymbol{b}^S_{n+1} = \boldsymbol{S} \times \frac{\partial}{\partial x}(\boldsymbol{b}^S_n) + (\boldsymbol{S} \times \boldsymbol{S}_x) \int_{\pm\infty}^{x} \langle \boldsymbol{b}^S_n, \boldsymbol{S}_x\rangle \,\mathrm{d}x + (K(\boldsymbol{c}_n))^S. \tag{55}$$

The problem is solved, but in order to put it in a more convenient form let us introduce the operator

$$\Lambda_\pm(X(x)) \equiv \boldsymbol{S} \times \frac{\partial}{\partial x}\boldsymbol{X}(x) + (\boldsymbol{S} \times \boldsymbol{S}_x) \int_{\pm\infty}^{x} \langle \boldsymbol{X}(x), \boldsymbol{S}_x\rangle \,\mathrm{d}x \tag{56}$$

$\boldsymbol{X}(x)$ being a vector field. Then we can formulate our results in the following proposition.

Proposition 3. The LL chain system has the following solution:

$$\left.\begin{aligned}&\boldsymbol{b}_0 = \boldsymbol{S} \qquad \boldsymbol{c}_0 = 0\\&\boldsymbol{b}_1 = \boldsymbol{S} \times \boldsymbol{S}_x \qquad \boldsymbol{c}_1 = K(\boldsymbol{S})\\&\boldsymbol{b}_{n+1} = \boldsymbol{b}^S_{n+1} + \boldsymbol{S}\int_{\pm\infty}^{x} \langle \boldsymbol{b}^S_{n+1}, \boldsymbol{S}_x\rangle \,\mathrm{d}x\\&\boldsymbol{b}^S_{n+1} = \Lambda_\pm(\boldsymbol{b}^S_n) + (K(\boldsymbol{c}_n))^S\\&\boldsymbol{c}_n = \sum_{q=1}^{n}(-1)^{q-1}K^{(q)}(\boldsymbol{b}_{n-q})\end{aligned}\right\} \qquad n = 1, 2, \ldots . \tag{57}$$

The operator $\Lambda_\pm$ is the so-called recursion or generating operator for the Heisenberg ferromagnet equation hierarchy of soliton equations. It was calculated for the first time in [13] (see also [14]), using the gauge equivalence between the Heisenberg ferromagnet equation and the nonlinear Schrödinger equation [16]. There are at least two other possible ways of arriving at this operator: geometrical (see, for example, [15]), or by solving the corresponding chain system for the Heisenberg ferromagnet equation hierarchy (see [10]). The Heisenberg ferromagnet (HF) equation is the following system:

$$\boldsymbol{S}_t = \boldsymbol{S} \times \boldsymbol{S}_{xx}. \tag{58}$$

Here $\boldsymbol{S}(x,t) = (S_1(x,t), S_2(x,t), S_3(x,t))$ is vector field depending on the spatial variable x and the time t, taking its values on the unit sphere $S^2 \subset \mathbb{R}^3$. The boundary conditions for this equation are similar to those for the LL equation:

$$\lim_{x\to\pm\infty} \boldsymbol{S} = (0,0,1). \tag{59}$$

Formally the HF equation is obtained from the LL equation if $r_i = 0$. Therefore it is natural to expect that when certain parameters (in our case j_i) tend to zero one can obtain the recursion scheme of HF from the recursion scheme in the LL case. As it is seen from the above proposition this is indeed the case in our approach. Surprisingly, for the elliptic bundle when the parameters r_i tend to zero one obtains not Λ, but Λ^2, see [9]. However, as we shall see below the hierarchies of equations obtained via elliptic and polynomial bundles seem to be equivalent in a sense to be described below.

Finally, let us write the first evolution equations from the hierarchy corresponding to the chain system solution that was given in proposition 3. These equations as mentioned are written in the form

$$\boldsymbol{S}_t = -\boldsymbol{S} \times b_{N+1} \qquad N = 0, 1, 2, \dots . \tag{60}$$

We have the following equations.

1. $N = 0$. The first equation in the hierarchy, as often happens, is linear:

$$\boldsymbol{S}_t = \boldsymbol{S}_x. \tag{61}$$

2. $N = 1$. The second equation of the hierarchy is

$$\boldsymbol{S}_t = (\boldsymbol{b}_1)_x - \boldsymbol{S} \times \boldsymbol{c}_1 = \boldsymbol{S} \times \boldsymbol{S}_{xx} - \boldsymbol{S} \times K^2(\boldsymbol{S}). \tag{62}$$

If we choose $j_i^2 = -r_i$; $i = 1, 2, 3$ we obtain the Landau–Lifshitz equation.
3. $N = 2$. The third equation of the hierarchy comprises

$$\boldsymbol{b}_2^S = \boldsymbol{S} \times (\boldsymbol{S} \times \boldsymbol{S}_{xx}) - \boldsymbol{S} \times (\boldsymbol{S} \times K^2\boldsymbol{S}) = (-\boldsymbol{S}_{xx} + K^2(\boldsymbol{S}))^S.$$

$$\begin{aligned}
\boldsymbol{b}_2 &= (-\boldsymbol{S}_{xx} + K^2(\boldsymbol{S}))^S + \boldsymbol{S} \int_{-\infty}^{x} \langle -\boldsymbol{S}_{xx} + K^2(\boldsymbol{S}), \boldsymbol{S}_x \rangle \, \mathrm{d}x \\
&= (-\boldsymbol{S}_{xx} + K^2(\boldsymbol{S}))^S + \tfrac{1}{2}\boldsymbol{S}(\langle K^2(\boldsymbol{S}), \boldsymbol{S}\rangle - \langle K^2(\boldsymbol{S}_0), \boldsymbol{S}_0\rangle) - \tfrac{1}{2}\boldsymbol{S}\langle \boldsymbol{S}_x, \boldsymbol{S}_x\rangle \\
&= -\boldsymbol{S}_{xx} + K^2(\boldsymbol{S}) + \boldsymbol{S}\left(\langle \boldsymbol{S}_{xx}, \boldsymbol{S}\rangle - \tfrac{1}{2}(\langle K^2(\boldsymbol{S}), \boldsymbol{S}\rangle - \langle K^2(\boldsymbol{S}_\mathbf{o}), \boldsymbol{S}_\mathbf{o}\rangle)\right) \\
&= -\boldsymbol{S}_{xx} + K^2(\boldsymbol{S}) - \boldsymbol{S}\left(\tfrac{3}{2}\langle \boldsymbol{S}_x, \boldsymbol{S}_x\rangle + \tfrac{1}{2}(K^2(\boldsymbol{S}), \boldsymbol{S}\rangle - K^2(\boldsymbol{S}_\mathbf{o}), \boldsymbol{S}_\mathbf{o}\rangle)\right).
\end{aligned}$$

The corresponding evolution equation is

$$\boldsymbol{S}_t = (\boldsymbol{b}_2)_x - \boldsymbol{S} \times K(\boldsymbol{c}_2) = (\boldsymbol{b}_2)_x - \boldsymbol{S} \times K(K(\boldsymbol{b}_1) + K^{(2)}(\boldsymbol{S})).$$

But, $KK^{(2)} = j_1 j_2 j_3 \mathbb{1}_3$ and therefore for the equation we have

$$\boldsymbol{S}_t = (\boldsymbol{b}_2)_x - \boldsymbol{S} \times K^2(\boldsymbol{S} \times \boldsymbol{S}_x).$$

We get

$$\begin{aligned}
\boldsymbol{S}_t = -\boldsymbol{S}_{xxx} - \boldsymbol{S} \times K^2(\boldsymbol{S} \times \boldsymbol{S}_x) + K^2(\boldsymbol{S}_x) - \boldsymbol{S}_x\left(\tfrac{3}{2}\langle \boldsymbol{S}_x, \boldsymbol{S}_x\rangle + \tfrac{1}{2}(\langle K^2(\boldsymbol{S}), \boldsymbol{S}\rangle\right. \\
\left. - \langle K^2(\boldsymbol{S}_0), \boldsymbol{S}_0\rangle)\right) - \boldsymbol{S}\left(3\langle \boldsymbol{S}_x, \boldsymbol{S}_{xx}\rangle + \langle K^2(\boldsymbol{S}), \boldsymbol{S}_x\rangle\right)
\end{aligned} \tag{63}$$

which after a brief calculation can be put in the final form

$$S_t = -S_{xxx} - 3S\langle S_x, S_{xx}\rangle - \tfrac{3}{2}S_x \left(\langle S_x, S_x\rangle + \langle K^2(S), S\rangle - \tfrac{2}{3}\operatorname{tr} K^2 - \tfrac{1}{3}\langle K^2(S_o), S_o\rangle\right). \quad (64)$$

If we accept $S_0 = (0, 0, 1)$ we get

$$S_t = -S_{xxx} - 3S\langle S_x, S_{xx}\rangle - \tfrac{3}{2}S_x \left(\langle S_x, S_x\rangle + \langle K^2(S), S\rangle - \tfrac{1}{3}(2j_1^2 + 2j_2^2 + 3j_3^2)\right). \quad (65)$$

The above equation differs from the next equation in the LL hierarchy found via the elliptic pairs (see, for example, [9]). Using our notation this equation can be written as follows:

$$S_t = S_{xxx} + 3S\langle S_x, S_{xx}\rangle + \tfrac{3}{2}S_x \left(\langle S_x, S_x\rangle + \langle K^2(S), S\rangle - j_3^2\right). \quad (66)$$

This equation was obtained by Date *et al* [17].

4. Discussion

As far as we know, the set of polynomial Lax pairs for the CF hierarchy has not been presented until now. Almost the same is true for the corresponding hierarchy of equations, because it is very difficult to obtain the corresponding hierarchy of soliton equations from the results of [11]. Therefore there is limited scope for comparing our result with those of others. For the LL case, however, the corresponding hierarchy of soliton equations obtained via elliptic bundle exists [9]. Let us briefly describe the situation of the LL hierarchies obtained via elliptic and via polynomial bundle. The first nonlinear evolution equations in both hierarchies coincide. It is simply the LL equation. The second nonlinear equations, however, are different. Nevertheless, one can say that up to the third equation both hierarchies are equivalent. Indeed, the equations in the hierarchies have the form

$$\left.\begin{aligned} S_t &= X_n(S, S_x, \ldots) \\ S_t &= Y_n(S, S_x, \ldots) \end{aligned}\right\} \qquad n = 1, 2, \ldots \quad (67)$$

their right-hand sides being vector fields on the infinite dimensional manifold of ‘potentials’, i.e. the set of functions $S(x)$. The hierarchies would be equivalent not only if $X_n = Y_n,\ n = 1, 2, \ldots$, but also in the case when every X_n is finite linear combination with constant coefficients of the fields Y_n. For example, if we denote by Y_n the fields obtained via polynomial bundle then the field corresponding to the equation of Date *et al* can be written as

$$-Y_3 + (j_1^2 + j_2^2)Y_1. \quad (68)$$

We believe that both hierarchies are equivalent in the sense mentioned above but, of course, the question of the equivalence of both hierarchies remains open.

We also leave to the future the questions about the Hamiltonian structures of the equations in LL and CF hierarchies and about the commutativity of the corresponding flows. These results will be published elsewhere.

Acknowledgments

One of the authors (ABY) would like to thank the Institute for Mathematics at the Leipzig University for their kind hospitality during his stay in 1995 when this paper was completed. This work was partially supported by the contract MM 428/94 with the Ministry of Science and Education of Bulgaria, by the DAAD (Germany) and by the NTZ of Leipzig University.

References

[1] Faddeev L D and Takhtadjan L A 1987 *Hamiltonian Method in the Theory of Solitons* (Berlin: Springer)
[2] Bordag L A and Yanovski A B 1995 *J. Phys. A: Math. Gen.* **28** 4007
[3] Landau L D and Lifshitz E M 1935 (published 1965 *Collected Papers of L D Landau* ed D ter Haar (New York: Pergamon, Gordon and Breach) p 101)
[4] Vesselov A P 1983 *Dokl. Akad. Nauk SSSSR* **270** 1094
[5] Vesselov A P 1984 *Dokl. Akad. Nauk SSSSR* **276** 590
[6] Borovik A E 1978 *Lett. JETP* **28** N10 629; **35** N7 3687-3723
[7] Sklianin E A 1979 *Steklov Math. Institute LOMI Preprint* E-3-79
[8] Cherednik I V 1981 *J. Nucl. Phys.* **33** 278
[9] Barouch E, Fokas A S and Papageorgiou V G 1987 *Clarkson University Preprint*
[10] Barouch E, Fokas A S and Papageorgiou V G 1988 *J. Math. Phys.* **29** 2628
[11] Sidorenko Yu N 1987 *Zap. LOMI* **161** 76
[12] Fuchssteiner B 1984 *Physica* **13D** 387
[13] Gerdjikov V S and Yanovski A B 1984 *Phys. Lett.* **103A** 232
[14] Gerdjikov V S and Yanovski A B 1984 *Commun. Math. Phys.* **103** N4 549
[15] Magri F, Morosi C and Ragnisco O 1985 *Commun. Math. Phys.* **99** 115
[16] Zakharov V E and Takhtadjan L A 1979 *Teor. Mat. Fiz.* **38** 26
[17] Date E, Jimbo M, Kashiwara M and Miwa T 1983 *J. Phys. A: Math. Gen.* **16** 221